Ekstremne vrijednosti funkcija više promjenljivih

54. Odrediti lokalni ekstremum funkcije $z = xy + \frac{50}{x} + \frac{20}{y}$, $x>0$, $y>0$

$$\frac{\partial z}{\partial x} = y - \frac{50}{x^2} \rightarrow \frac{\partial z}{\partial x} = 0 \Rightarrow y - \frac{50}{x^2} = 0$$

$$\frac{\partial z}{\partial y} = x - \frac{20}{y^2} \rightarrow \frac{\partial z}{\partial y} = 0 \Rightarrow x - \frac{20}{y^2} = 0 \rightarrow x = \frac{20}{y^2}$$

uvrstimo u j-nu $\frac{\partial z}{\partial x} = 0$

$$y - \frac{50}{\left(\frac{20}{y^2}\right)^2} = 0 \Rightarrow y - \frac{50}{400} y^4 = 0 \rightarrow y - \frac{y^4}{8} = 0$$

$$y - \frac{y^4}{8} = y\left(1 - \frac{y^3}{8}\right) = 0 \Leftrightarrow y=0 \lor y=2$$

$y=0 \notin D \Rightarrow y=2$, $x = \frac{20}{4} = 5 \rightarrow A(5,2)$ je stacionarna tačka.

$$\frac{\partial^2 z}{\partial x^2} = \frac{100}{x^3} \qquad \boxed{\frac{\partial^2 z}{\partial x^2}(A) = \frac{100}{125} = \frac{4}{5}} \qquad \boxed{\frac{\partial^2 z}{\partial x \partial y} = 1}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{40}{y^3} \qquad \boxed{\frac{\partial^2 z}{\partial y^2}(A) = \frac{40}{8} = 5}$$

1° $\Delta_1 = \frac{\partial^2 z}{\partial x^2}(A) = \frac{100}{125} > 0$

2° $$\Delta_2 = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2}(A) & \frac{\partial^2 z}{\partial x \partial y}(A) \\ \frac{\partial^2 z}{\partial y \partial x}(A) & \frac{\partial^2 z}{\partial y^2}(A) \end{vmatrix} = \begin{vmatrix} \frac{4}{5} & 1 \\ 1 & 5 \end{vmatrix} = \frac{4}{5} \cdot 5 - 1 = 3 > 0$$

iz 1° i 2° slijedi da je $d^2 z(A) > 0$ pa fja u tački A postiže lokalni minimum

$$z_{min} = z(A) = z(5,2) = 5 \cdot 2 + \frac{50}{5} + \frac{20}{2} = 30$$

55. Odrediti lokalne ekstremume funkcije:

$$z = 2\cdot(x-2y)^2 + x^3 - 2y^3 + 1$$

R//

$$\frac{\partial z}{\partial x} = 4(x-2y) + 3x^2$$

$$\frac{\partial z}{\partial y} = 4(x-2y)\cdot(-2) - 6y^2$$

$$\frac{\partial z}{\partial x} = 0 \Rightarrow 4(x-2y) + 3x^2 = 0 \quad \Big/ \cdot 2$$

$$-8(x-2y) - 6y^2 = 0$$

Riješiti sistem najlakše

$$4(x-2y) + 3x^2 = 0$$

$$6x^2 - 6y^2 = 0 \rightarrow \boxed{x^2 = y^2}$$

1° $x = y$

$$\Rightarrow -4y + 3y^2 = 0 \rightarrow y(-4+3y) = 0 \Leftrightarrow y = 0 \vee y = \frac{4}{3}$$

$$y = 0 \Rightarrow x = 0 \rightarrow \boxed{A(0,0)}; \quad y = \frac{4}{3} \Rightarrow x = \frac{4}{3} \Rightarrow \boxed{B\left(\frac{4}{3}, \frac{4}{3}\right)}$$

2° $x = -y$

$$\Rightarrow 4(-y-2y) + 3y^2 = 0 \rightarrow -12y + 3y^2 = 0$$

$$\rightarrow y(-12+3y) = 0 \Leftrightarrow y = 0 \vee y = 4$$

$A(0,0)$ $\Leftrightarrow$ $x = -4 \rightarrow \boxed{C(-4,4)}$

$$\frac{\partial^2 z}{\partial x^2} = 4 + 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -8$$

$$\frac{\partial^2 z}{\partial y^2} = +16 - 12y$$

$\boxed{\text{I} \quad A(0,0)}$

$$1° \ \Delta_1 = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}(0,0) = 4 > 0$$

$$2° \ \Delta_2 = \begin{vmatrix} \frac{\partial^2 z}{\partial x^2}(A) & \frac{\partial^2 z}{\partial x \partial y}(A) \\ \frac{\partial^2 z}{\partial y \partial x}(A) & \frac{\partial^2 z}{\partial y^2}(A) \end{vmatrix} = \begin{vmatrix} 4 & -8 \\ -8 & 16 \end{vmatrix} = 64 - 64 = 0$$

$\rightarrow$ o tački A ništa ne možemo zaključiti,

$\nexists$ ne možemo ništa zaključiti o $d^2 z(A)$

$\rightarrow$ moramo drugačije:

Posmatrajmo priraštaj funkcije:

$$\Delta f = f(0+\Delta x, 0+\Delta y) - f(0,0) =$$

$$= f(\Delta x, \Delta y) - f(0,0) = 2(\Delta x - 2\Delta y)^2 + \Delta x^3 \cancel{+1} - 1$$

$$= 2(\Delta x - 2\Delta y)^2 + \Delta x^3 - 2\Delta y^3$$

$f(0,0)$

neka je 1° $\Delta x = 2t$, $\Delta y = t$, $t > 0$

dobijamo da je $\Delta f = 8t^3 - 2t^3 = 6t^3 > 0$

2° za $\Delta x = 2t$, $\Delta y = t$, $t < 0$

dobijamo da je $\Delta f = 6t^3 < 0$

$\rightarrow$ iz 1° i 2° $\Rightarrow$ da u okolini tačke A priraštaj fje mijenja znak u zavisnosti od znaka priraštaja argumenta pa u tački A funkcija ne postiže lokalni ekstremum.

II $\boxed{B\left(\frac{4}{3}, \frac{4}{3}\right)}$

1° $\Delta_1 = \frac{\partial^2 z}{\partial x^2}(B) = 4 + \cancel{6}^2 \cdot \frac{4}{\cancel{3}_1} = 4 + 8 = 12 > 0$

$$\frac{\partial^2 z}{\partial x^2} = 4 + 6x$$

$$\frac{\partial^2 z}{\partial y^2} = 16 - 12y$$

$$\frac{\partial^2 z}{\partial x \partial y} = -8$$

$16 - \cancel{12}^4 \cdot \frac{4}{\cancel{3}_1} = 16 - 16 = 0$

2°
$$\Delta_2 = \begin{vmatrix} 12 & -8 \\ -8 & 0 \end{vmatrix} = -64 < 0$$

Kad imamo da je $\Delta_2 < 0$ $\Rightarrow$ $\boxed{\text{sjedlo}}$

$1_7\ \Delta_1$ i Δ_2 slijedi

$C(-4,4)$

$\Delta_1(-4,4) = 4-24 = -20 < 0$ $\Rightarrow$ tačka C je lokalni maksimum

$\Delta_2(-4,4) = \begin{vmatrix} -20 & -8 \\ -8 & -32 \end{vmatrix} > 0$

(56.) Naći ekstremne vrijednosti funkcije:

$f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$

R//

$\frac{\partial f}{\partial x} = 3x^2 + 12y \rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 12y = 0$

$x^2 + 4y = 0$

$\frac{\partial f}{\partial y} = 2y + 12x \rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow y + 6x = 0$

$y = -6x$

$\frac{\partial f}{\partial z} = 2z + 2 \rightarrow \frac{\partial f}{\partial z} = 0 \Rightarrow \boxed{z = -1}$

$\longrightarrow y = -6x$

$\cancel{y}\ x^2 + 4\cdot(-6x) = 0 \rightarrow x^2 - 24x = 0$

$x(x-24) = 0$

$\Leftrightarrow x = 0 \lor x = 24$

$y = 0 \qquad y = -144$

$\boxed{A(0,0,-1)}$ $\boxed{B(24,-144,-1)}$ $\rightarrow$ stacionarne tačke

$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial x \partial y} = 12$

$\frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial z} = 0$

$\frac{\partial^2 f}{\partial z^2} = 2 \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$

	max	min
Δ_1	<	>
Δ_2	>	>
Δ_3	<	>
Δ_4	<	>

$\boxed{\text{I} \quad A(0,0,-1)}$

$$\Delta_1 = \frac{\partial^2 f}{\partial x^2}(A) = 6 \cdot 0 = 0$$

Dodatno diskutujemo:

$$\Delta f = f(0+\Delta x, 0+\Delta y, -1+\Delta z) - f(0,0,-1) =$$

$$= f(\Delta x, \Delta y, \Delta z - 1) - f(0,0,-1) =$$

$$= \Delta x^3 + \Delta y^2 + (\Delta z - 1)^2 + 12\,\Delta x \Delta y + 2(\Delta z - 1) - (-1)$$

a) za $\Delta z = 0$

$\Delta y = 0$

$\Delta x = t, \; t > 0$

dobijamo da je $\Delta f = (-1)^2 + 2(0-1) + 1 + t^3$

$$= 2 - 2 + \cancel{t^3} - t^3 > 0 \quad 2)$$

b) za $\Delta z = 0$

$\Delta y = 0 \qquad \Delta f = t^3 < 0 \quad 1)$

$\Delta x = t, \; t < 0$

iz ~~t~~ 1) i 2) $\Rightarrow$ tačka $(0,0,-1)$ nije lokalni ekstremum jer oko tačke A fja mijenja znak

$\text{II} \quad \boxed{B = (24, -144, -1)}$

1) $\Delta_1 = \frac{\partial^2 f}{\partial x^2}(B) = 6 \cdot 24 = 144 > 0 \quad (1)$

2) $$\Delta_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(B) & \frac{\partial^2 f}{\partial x \partial y}(B) \\ \frac{\partial^2 f}{\partial y \partial x}(B) & \frac{\partial^2 f}{\partial y^2}(B) \end{vmatrix} = \begin{vmatrix} 144 & 12 \\ 12 & 2 \end{vmatrix} =$$

$$= 288 - 144 = 144 > 0 \quad (2)$$

$F(x,y,z)$

3) $$\Delta_3 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(B) & \frac{\partial^2 f}{\partial x \partial y}(B) & \frac{\partial^2 f}{\partial x \partial z}(B) \\ \frac{\partial^2 f}{\partial y \partial x}(B) & \frac{\partial^2 f}{\partial y^2}(B) & \frac{\partial^2 f}{\partial y \partial z}(B) \\ \frac{\partial^2 f}{\partial z \partial x}(B) & \frac{\partial^2 f}{\partial z \partial y}(B) & \frac{\partial^2 f}{\partial z^2}(B) \end{vmatrix} =$$

$$= \begin{vmatrix} 144 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 4 \cdot 144 - 2 \cdot 144 = 2 \cdot 144 > 0 \quad (3)$$

iz (1), (2) i (3) slijedi da je $d^2 f(B) > 0$ pa fja f u tački B postiže lokalni minimum

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57. Odrediti ekstremne vrijednosti funkcije:

$$z(x,y) = x^3 + y^3 - 3xy$$

R// $$\frac{\partial z}{\partial x} = 3x^2 - 3y \rightarrow \frac{\partial z}{\partial x} = 0 \Leftrightarrow 3x^2 - 3y = 0 \Leftrightarrow x^2 = y$$

$$\frac{\partial z}{\partial y} = 3y^2 - 3x \rightarrow \frac{\partial z}{\partial y} = 0 \Leftrightarrow 3y^2 - 3y = 0 \Leftrightarrow y^2 = x$$

$y^2 = x \Rightarrow$ uvrstimo u jednačinu $x^2 - y = 0$

$$y^4 - y = 0 \Leftrightarrow y(y^3 - 1) = 0 \Leftrightarrow y = 1 \vee y = 0$$

$$x = 1 \qquad x = 0$$

$\rightarrow$ stacionarne tačke:

$$A(0,0) \quad ; \quad B(1,1)$$

AGATHA RUIZ DE LA PRADA

$$\rightarrow \frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -3$$

$$\frac{\partial^2 z}{\partial y^2} = 6y$$

$$d^2 z = \frac{\partial^2 z}{\partial x^2} \cdot dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \cdot dx\, dy + \frac{\partial^2 z}{\partial y^2} dy^2$$

$\boxed{\text{I} \quad A(0,0)}$

$$d^2 z\,(0,0) = 0 \cdot dx^2 + (-6)\, dx\, dy + 0 \cdot dy^2$$

$$d^2 z\,(0,0) = -6\, dx\, dy$$

1° $dx > 0$, $dy < 0$ $\quad d^2 z > 0$ $\qquad$ 2° $dx < 0$, $dy < 0$ $\quad d^2 z < 0$

$\rightarrow$ iz 1° i 2° slijedi da je diferencijal promjenljivog znaka $\Rightarrow$ u $A(0,0)$ fja ne postiže ekstremum

$\boxed{\text{II} \quad B(1,1)}$

$$d^2 z(1,1) = 6dx^2 - 6dx\,dy + 6dy^2 =$$

$$= \underbrace{3dx^2 + 3dy^2}_{\geq 0} + 3\underbrace{(dx - dy)^2}_{\geq 0}$$

$$d^2 z\,(1,1) > 0 \Rightarrow |dx| + |dy| \neq 0 \quad \begin{pmatrix} dx \neq 0 \vee \\ dy \neq 0 \end{pmatrix}$$

$\Rightarrow$ u tački $B(1,1)$ fja z postiže

lokalni minimum.

58) Naći ekstremne vrijednosti funkcije:

$$z = 2x - 2y + \ln(2x - x^2 - y^2)$$

→ domen → razlomak, $\overset{\text{paran}}{\text{korijen}}$, ln

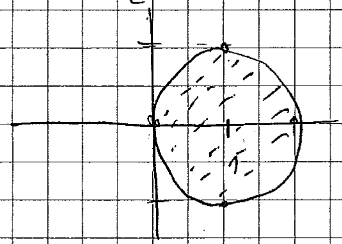
1) Odredimo oblast definisanosti

$$2x - x^2 - y^2 > 0 \rightarrow x^2 - 2x + 1 + y^2 - 1 < 0$$

$$x^2 + y^2 - 2x < 0 \qquad (x-1)^2 + y^2 < 0$$

Oblast definisanosti je:

$$D = \{x, y \mid (x-1)^2 + y^2 < 1\} \rightarrow \text{j-na kruga } C(1,0),\ r=1$$



→ Domen krug bez spoljašnjosti

2) Stacionarne tačke → preko 1. izvoda

$$\frac{\partial z}{\partial x} = 2 + \frac{1}{2x - x^2 - y^2} \cdot (2 - 2x)$$

$$\frac{\partial z}{\partial y} = 2 + \frac{1}{2x - x^2 - y^2} \cdot (-2y)$$

$$\frac{\partial z}{\partial x} = 0 \rightarrow 1 + \frac{1 - x}{2x - x^2 - y^2} = 0 \quad / \cdot \boxed{(2x - x^2 - y^2)}$$

MOŽEMO JER JE OBLAST DEFINISANOSTI >0.

$$\frac{\partial z}{\partial y} = 0 \rightarrow -1 - \frac{y}{2x - x^2 - y^2} = 0 \quad / \cdot (2x - x^2 - y^2)$$

$$\begin{cases} 2x - x^2 - y^2 + 1 - x = 0 \\ -2x + x^2 + y^2 - y = 0 \end{cases} +$$

$$x - x^2 - y^2 + 1 = 0$$

$$-x - y + 1 = 0 \longrightarrow y = 1 - x$$

$-\frac{8}{4} = -2$

$-\frac{10}{4}$

$x - x^2 - 1 + 2x - x^2 + 1 = 0$

$3x - 2x^2 = 0$

$x(3-2x) = 0 \iff x=0 \quad \vee \quad x = \frac{3}{2}$

$y = 1 \rightarrow \boxed{A(0,1)}$ $\qquad y = -\frac{1}{2}$ $\boxed{B\left(\frac{3}{2}, -\frac{1}{2}\right)}$

$A \notin D \Rightarrow A$ nije stacionarna tačka

$B \in D \Rightarrow B$ je stacionarna tačka

$$\frac{\partial^2 z}{\partial x^2} = 2 \cdot \frac{-x^2+y^2+2x-2}{(2x-x^2-y^2)^2}$$

-8

$-\frac{8}{4}$ ~~…~~ $+2\cdot\frac{3}{2} - 2 =$

$$\frac{\partial^2 z}{\partial y^2} = -2\frac{2x-x^2+y^2}{(2x-x^2-y^2)^2}$$

$= -2+3-2 = -4+3 = -1$

-2

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{2y(2x-2)}{(2x-x^2-y^2)^2}$$

$B\left(\frac{3}{2}, -\frac{1}{2}\right)$

$$\Delta_1 = \frac{\partial^2 z}{\partial x^2}(B) = 2 \cdot \frac{-\frac{9}{4}+\frac{1}{4}+2\cdot\frac{3}{2}-2}{\left(2\cdot\frac{3}{2}-\frac{9}{4}-\frac{1}{4}\right)^2} = 2\cdot\frac{-\frac{8}{4}+\frac{4}{4}}{\left(3-\frac{10}{4}\right)^2} =$$

$-2+1$

$$= \frac{-2}{\left(\frac{12-10}{4}\right)^2} = \frac{-2}{\left(\frac{2}{4}\right)^2} = -8 < 0 \quad 1^\circ$$

2°

$$\Delta_2 = \begin{vmatrix} -8 & -4 \\ -4 & -8 \end{vmatrix} = 48 > 0$$

$\rightarrow$ iz 1° i 2° slijedi da je $d^2 z(B) < 0$ pa ~~je~~ fja z u tački $B\left(\frac{3}{2}, -\frac{1}{2}\right)$ postiže lokalni MAKSIMUM

$\Delta_1 > 0$ $\quad\quad$ $\Delta_1 < 0$

$\Delta_2 > 0$ $\quad\quad$ $\Delta_2 > 0$ $\quad\quad$ $\Delta_2 < 0 \Rightarrow$ nije extremna vrijednost u toj tački

$\Delta_3 > 0$ $\quad\quad$ $\Delta_3 < 0$

$d^2 f > 0 \Rightarrow \min$ $\quad\quad$ $d^2 f < 0 \Rightarrow \text{Max}$

59. Naći ekstremne vrijednosti funkcije $z = x + y$ uz uslov $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{4}$

Formirajmo Lagranžovu funkciju:

$$\boxed{\mathcal{L}(x, y, \lambda) = z(x, y) + \lambda \cdot g(x, y),}$$ gdje je

$g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{4}$ a uslov je $g(x,y) = 0$

$$\mathcal{L}(x, y, \lambda) = x + y + \lambda \cdot \left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{4}\right)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - \frac{2\lambda}{x^3} \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{4}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - \frac{2\lambda}{y^3}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow 1 - \frac{2\lambda}{x^3} = 0 \quad 1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow 1 - \frac{2\lambda}{y^3} = 0 \quad 2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{4} = 0 \quad 3)$$

iz 1) $\Rightarrow x^3 = 2\lambda \Rightarrow x = \sqrt[3]{2\lambda}$

iz 2) $\Rightarrow y^3 = 2\lambda \Rightarrow y = \sqrt[3]{2\lambda}$ $\Rightarrow x^3 = y^3 \Rightarrow \boxed{x = y}$

$$\frac{1}{x^2} + \frac{1}{x^2} - \frac{1}{4} = 0 \longrightarrow \frac{2}{x^2} = \frac{1}{4} \to x^2 = 8$$

$\boxed{x = \pm\sqrt{2}}$ $\boxed{A(2\sqrt{2}, 2\sqrt{2})}$ $\boxed{B(-2\sqrt{2}, -2\sqrt{2})}$

za $x = 2\sqrt{2} \Rightarrow \lambda = \frac{x^3}{2} = 8\sqrt{2}$

$x = -2\sqrt{2} \Rightarrow \lambda = \frac{x^3}{2} = -8\sqrt{2}$

$$\frac{\partial^2 L}{\partial x^2} = \frac{6\lambda}{x^4} \qquad \frac{\partial^2 L}{\partial y^2} = \frac{6\lambda}{x^4} \qquad \frac{\partial^2 L}{\partial x \partial y} = 0$$

$$d^2 L = \frac{\partial^2 L}{\partial x^2} dx^2 + 2\frac{\partial^2 L}{\partial x \partial y} \cdot dx\,dy + \frac{\partial^2 L}{\partial y^2} dy^2$$

$\longrightarrow$ uslov $dg = 0$

pretpostavka $|dx| + |dy| \neq 0$

1) $d^2 L = \frac{6\lambda}{x^4} dx^2 + \frac{6\lambda}{y^4} dy^2$

2) uslov:

$$\frac{\partial g}{\partial x} \cdot dx + \frac{\partial g}{\partial y} \cdot dy = 0$$

3) pretpostavka $|dx| + |dy| \neq 0$

I $A(2\sqrt{2}, 2\sqrt{2})$ $\lambda = 8\sqrt{2}$

$\longrightarrow$ odredimo kojeg je znaka L uz uslov

(*) $d^2 L(A) = \frac{6\cdot 8\sqrt{2}}{64} dx^2 + \frac{6\cdot 8\sqrt{2}}{64} dy^2 = \frac{3\sqrt{2}}{4}(dx^2+dy^2)$

uslov

$$\frac{2}{16\sqrt{2}} \cdot dx + \frac{2}{16\sqrt{2}} dy = 0$$

$$dx + dy = 0$$

$\boxed{dx = -dy} \rightarrow$ ovaj uslov vratimo u j-nu (*)

+ pretpostavka $dx \neq 0 \vee dy \neq 0$

$$d^2 L(A) = \frac{3\sqrt{2}}{4} 2dy^2 > 0$$

$$dy \neq 0$$

napomena: $dy = 0 \Rightarrow dx = 0 \rightarrow$ a to ne može

$\rightarrow$ funkcija postiže uslovni minimum u tački A

II $B(-2\sqrt{2}, -2\sqrt{2})$, $\lambda = -8\sqrt{2}$

$$d^2 L(B) = 6 \cdot (-8\sqrt{2}) \cdot \left(\frac{1}{64} dx^2 + \frac{1}{64} dy^2\right) =$$

$$= -\frac{3\sqrt{2}}{4}(dx^2 + dy^2)$$

uslov:

$$\frac{2}{-16\sqrt{2}} \cdot (dx + dy) = 0 \rightarrow dx = -dy$$

pretp. $dx \neq 0 \vee dy \neq 0$

uvrstimo u j-nu $d^2L(B)$

$d^2 L = -\frac{3\sqrt{2}}{4} \cdot 2dy^2 < 0 \rightarrow dy \neq 0$ (da je $dy = 0 \Rightarrow dx = 0$ a to ne može)

$\Rightarrow d^2L$ u (B) ~~po~~ $< 0 \Rightarrow$ fja 2 u tački B postiže USLOVNI maksimum

60 Odrediti tačke elipsoida $\frac{x^2}{8}+\frac{y^2}{4}+\frac{z^2}{2}=1$ koje su najmanje i najviše udaljene od tačke $M(0,0,3)$

R// Neka je $T(x,y,z)$ proizvoljna tačka elipsoida

$$d(T,M)=\sqrt{x^2+y^2+(z-3)^2} \quad (*)$$

→ tražimo najmanju vrijednost f-je (*) uz uslov da tačka pripada elipsoidu, tj. da zadovoljava jednačinu:

$$\frac{x^2}{8}+\frac{y^2}{4}+\frac{z^2}{2}=1$$

→ Posmatraćemo (zbog lakšeg računa) f-ju:

$$f(x,y,z)=x^2+y^2+(z-3)^2 \text{ uz uslov } \frac{x^2}{8}+\frac{y^2}{4}+\frac{z^2}{2}=1$$

→ Formirajmo Lagranžovu funkciju:

$$L(x,y,\lambda)=x^2+y^2+(z-3)^2+\lambda\left(x^2+2y^2+4z^2-8\right)$$

uslov → $g(x,y,z)=0$, $g(x,y,z)=x^2+2y^2+4z^2-8$

$$\frac{\partial L}{\partial x}=2x+2\lambda x \to \frac{\partial L}{\partial x}=0 \Rightarrow 2x+2\lambda x=0 \to x(1+\lambda)=0$$

$$\Rightarrow (1)\ \boxed{x=0 \lor \lambda=-1}$$

$$\frac{\partial L}{\partial y}=2y+4\lambda y \to \frac{\partial L}{\partial y}=0 \Rightarrow y+2\lambda y=0 \to y(1+2\lambda)=0$$

$$\Rightarrow (2)\ \boxed{y=0 \lor \lambda=-\frac{1}{2}}$$

$$\frac{\partial L}{\partial z}=2(z-3)+8\lambda z \to \frac{\partial L}{\partial z}=0 \Rightarrow 2z-6+8\lambda z=0$$

$$2z(1+4\lambda)=6 \to z(1+4\lambda)=3$$

$$(3)\ \boxed{z=\frac{3}{1+4\lambda}}$$

$$\frac{\partial L}{\partial \lambda}=x^2+2y^2+4z^2-8$$

$$\to \frac{\partial L}{\partial \lambda}=0 \Rightarrow x^2+2y^2+4z^2-8=0$$

→ Kombinuju se ① i ② uslovi

1° $\boxed{x=0 \wedge y=0}$

$4z^2 - 8 = 0$ → uvrstili u uslov

$z^2 = 2 \to z = \pm\sqrt{2}$

$\boxed{A(0,0,\sqrt{2})}$ $\boxed{B(0,0,-\sqrt{2})}$

A: $z = \dfrac{3}{1+4\lambda} \to$

$\sqrt{2} = \dfrac{3}{1-4\lambda} \to \sqrt{2} - 4\sqrt{2}\lambda = 3$

$\sqrt{2}(1+4\lambda) = 3 \to 1+4\lambda = \dfrac{3}{\sqrt{2}}$

$4\lambda = \dfrac{3\sqrt{2}}{2} - 1 = \dfrac{3\sqrt{2}-2}{2} \;\Big/ : 4$

$\boxed{\lambda_A = \dfrac{3\sqrt{2}-2}{8}}$

B: $z = -\sqrt{2} \to 1+4\lambda = -\dfrac{3\sqrt{2}}{2}$

$4\lambda = \dfrac{-3\sqrt{2}-2}{2}$

$\boxed{\lambda_B = \dfrac{-3\sqrt{2}-2}{8}}$

2° $\boxed{x=0 \wedge \lambda = -\frac{1}{2}}$ $\times$

$z = \dfrac{3}{1+4\left(-\frac{1}{2}\right)} = \dfrac{3}{-1} = -3$

$0 + 2y^2 + 4\cdot 9 - 8 = 0 \to 2y^2 = -26 \perp$

3° $\boxed{\lambda = -1 \wedge \lambda = -\frac{1}{2}}$ $\times$

4° $\boxed{\lambda = -1 \wedge y = 0}$

$z = \dfrac{3}{1+4\lambda} = \dfrac{3}{1-4} = -1$

$x^2 + 2\cdot 0 + 4\cdot 1 - 8 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\boxed{C(2,0,-1)}$ $\lambda_C = -1$

$\boxed{D(-2,0,-1)}$ $\lambda_D = -1$

$A(0,0,\sqrt{2})$; $\lambda_A = \frac{3\sqrt{2}-2}{8}$

$B(0,0,-\sqrt{2})$; $\lambda_B = \frac{-3\sqrt{2}-2}{8}$

$C(2,0,-1)$; $\lambda_C = -1$

$D(-2,0,-1)$; $\lambda_D = -1$

→ ispitujemo sve 4 tačke

$$\frac{\partial^2 L}{\partial x^2} = 2+2\lambda \qquad \frac{\partial^2 L}{\partial x \partial y} = 0 \qquad \frac{\partial^2 L}{\partial x \partial z} = 0$$

$$\frac{\partial^2 L}{\partial y^2} = 2+4\lambda \qquad \frac{\partial^2 L}{\partial z^2} = 2+8\lambda \qquad \frac{\partial^2 L}{\partial y \partial z} = 0$$

$$d^2L = \frac{\partial^2 L}{\partial x^2} dx^2 + \underbrace{2\frac{\partial^2 L}{\partial x \partial y} dx dy}_{0} + \underbrace{2\frac{\partial^2 L}{\partial x \partial z} dx dz}_{0} + \underbrace{2\frac{\partial^2 L}{\partial y \partial z} dy\, dz}_{0} + \frac{\partial^2 L}{\partial y^2} dy^2 + \frac{\partial^2 L}{\partial z^2} dz^2$$

1) $d^2L = 2(1+\lambda)dx^2 + (2+4\lambda)dy^2 + 2(1+4\lambda)dz^2$

2) uslov $dg = 0$

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz = 0$$

$$\frac{\partial g}{\partial x} = 2x \ ; \quad \frac{\partial g}{\partial y} = 4y \ ; \quad \frac{\partial g}{\partial z} = 8z$$

$$\rightarrow dg = 2x\,dx + 4y\,dy + 8z\,dz = 0$$

$$x\,dx + 2y\,dy + 4z\,dz = 0$$

3) pretpostavka $dx \neq 0 \vee dy \neq 0 \vee dz \neq 0$

$$\boxed{\text{I} \quad A\left(0,0,\sqrt{2}\right) \; ; \; \lambda_A = \frac{3\sqrt{2}-2}{8}}$$

$$d^2L(A) = 2\left(1+\frac{3\sqrt{2}-2}{8}\right)dx^2 + 2\left(1+2\cdot\frac{3\sqrt{2}-2}{8}\right)dy^2 +$$

$$+ 2\left(1+4\cdot\frac{3\sqrt{2}-2}{8}\right)dz^2 =$$

$$= \frac{6+3\sqrt{2}}{4}dx^2 + \frac{2+3\sqrt{2}}{2}dy^2 + 3\sqrt{2}\,dz^2$$

uslov $dg=0$

$$0\cdot dx + 2\cdot 0\cdot dy + 4\cdot\sqrt{2}\,dz = 0$$

$$4\sqrt{2}\,dz = 0 \rightarrow dz = 0$$

$$\Rightarrow d^2L = \frac{6+3\sqrt{2}}{4}dx^2 + \frac{2+3\sqrt{2}}{2}dy^2 \longrightarrow dx \neq 0 \vee dy \neq 0$$

($dx=0 \wedge dy=0$ a već $dz=0$ → nemoguće)

$$\boxed{d^2L > 0}$$

$d^2L(A) > 0 \Rightarrow f(x,y,z) = x^2+y^2+(z-3)^2$ postiže u A uslovni minimum pa i fja $g(x,y,z) = \sqrt{x^2+y^2+(z-3)^2}$ postiže u A uslovni minimum

$$\boxed{\text{II} \quad B\left(0,0,-\sqrt{2}\right) \; ; \; \lambda_B = \frac{-3\sqrt{2}-2}{8}}$$

$$d^2L(B) = 2\left(1+\frac{-3\sqrt{2}-2}{8}\right)dx^2 + 2\left(1+2\cdot\frac{-3\sqrt{2}-2}{8}\right)dy^2 +$$

$$+ 2\left(1+4\cdot\frac{-3\sqrt{2}-2}{8}\right)dz^2 =$$

$$= \frac{-3\sqrt{2}+6}{4}dx^2 + \frac{-3\sqrt{2}+2}{2}dy^2 - 3\sqrt{2}\,dz^2$$

uslov $dg=0$

$$0\,dx + 2\cdot 0\,dy^2 + 4\left(-\sqrt{2}\right)dz = 0$$

$$-4\sqrt{2}\,dz = 0 \rightarrow dz = 0 \quad \text{uvrstimo u } d^2L(B)$$

AGATHA RUIZ DE LA PRADA

$$d^2L(B) = \frac{6-3\sqrt{2}}{4}\, dx^2 + \frac{2-3\sqrt{2}}{2}\, dy^2$$

1° $dx = t > 0$, $dy = 0$

$$d^2L(B) = \frac{6-3\sqrt{2}}{4} \cdot t^2 > 0$$

2° $dx = 0$, $dy = t$, $t > 0$

$$d^2L(B) = \frac{2-3\sqrt{2}}{2} \cdot t^2 < 0$$

→ iz 1° i 2° slijedi da fja f, odnosno fja g u tački B ne postiže uslovni ekstremum

→ I ovdje možemo koristiti Silvesterov kriterijum ali paziti, za funkciju L !

$\boxed{\text{III} \quad C(2,0,-1);\ \lambda_c = -1}$

$$d^2L(C) = -2dy^2 - 6dz^2 = -2(dy^2 + 3dz^2)$$

→ ovdje možemo odmah ali ipak uslov

$dg = 0$ uslov

uslov: $2dx - 4dz = 0$

$dx = 2dz$ → uvrstimo u $d^2L(C)$

→ $d^2L(C) = -2(dy^2 + 3dz^2) < 0$, $dy \neq 0 \vee dz \neq 0$

$(dy = 0 \wedge dz = 0 \Rightarrow dx = 0)$

$d^2L(C) < 0 \Rightarrow$ fja f tj. fja g u tački C postiže uslovni maksimum

$\boxed{\text{IV} \quad D(-2,0,-1);\ \lambda_D = -1}$ → uslovni maksimum

$$d^2L(D) = -2dy^2 - 6dz^2 = -2(dy^2 + 3dz^2)$$

slično kao za C → uslov $dx = -2dz$

→ Tačka $A(0,0,\sqrt{2})$ je tačka sa elipsoida koja je najbliža tački $M(0,0,3)$. Udaljenost između A i M je: $d(A,M) = \sqrt{0^2+0^2+(\sqrt{2}-3)^2} = 3-\sqrt{2} = g_{min}$

→ Tačke $C(2,0,-1)$ i $D(-2,0,-1)$ su tačke sa elipsoide koje su najviše udaljene od tačke M

$d(C,M) = d(D,M) = g_{max} = g(2,0,-1) = \sqrt{20}$

---

61. Naći ekstremnu vrijednost proizvoda 3 pozitivna ~~redna~~ realna broja x, y i z ako je njihov zbir konstantan i iznosi $3c$

R// Neka je $f(x,y,z) = x\cdot y\cdot z$

$$g(x,y,z) = x+y+z-3c$$

→ tražimo najveću i najmanju vrijednost funkcije uz uslov $g(x,y,z)=0$, odnosno $x+y+z-3c=0$

→ Može Lagranžova funkcija ali lakše neku od promjenljivih da izrazimo i tražimo max/min fje 2 promjenljive

→ Iz uslova dobijamo: $z = 3c-x-y$

Iskoristićemo uslov u funkciji f

$$h(x,y) = f(x,y,3c-x-y) = \underbrace{x\cdot y}\cdot\underbrace{(3c-x-y)}$$

→ Tražimo globalni ekstremum funkcije $h(x,y)$

$$\frac{\partial h}{\partial x} = 0 \Rightarrow y\cdot(3c-x-y) + xy(-1) = 0$$

$$\frac{\partial h}{\partial y} = 0 \Rightarrow x\cdot(3c-x-y) + xy(-1) = 0$$

AGATHA RUIZ DE LA PRADA

$y\,(3c-2x-y)=0 \Rightarrow 3c-2x-y=0 \quad \Big\downarrow (-2)$

$x\,(3c-x-2y)=0 \Rightarrow 3c-x-2y=0$

$\boxed{\text{dato } x,y,z>0}$

$-3c+3x=0 \rightarrow \boxed{x=c}$

~~3c-6c+y+...~~ $3c-c-2y=0$

$\rightarrow \boxed{y=c}$

$\boxed{A\,(c,c)}$

$$\frac{\partial^2 h}{\partial x^2}=-2y \qquad \frac{\partial^2 h}{\partial y^2}=-2x \qquad \frac{\partial^2 h}{\partial x\partial y}=3c-2x-2y$$

$1^\circ \quad \Delta_1=\frac{\partial^2 h}{\partial x^2}(A)=-2c<0$

$c>0$

$$2^\circ \quad \Delta_2=\begin{vmatrix}\frac{\partial^2 h}{\partial x^2}(A) & \frac{\partial^2 h}{\partial x\partial y}(A)\\ \frac{\partial^2 h}{\partial y\partial x}(A) & \frac{\partial^2 h}{\partial y^2}(A)\end{vmatrix}=\begin{vmatrix}-2c & -c\\ -c & -2c\end{vmatrix}=3c^2>0$$

$\rightarrow$ iz 1° i 2° slijedi da je $d^2h(A)<0$ pa fja h u tački $A(c,c)$ postiže maksimum

$\rightarrow$ za funkciju f uočavamo tačku:

$$B\,(c,\,c,\,3c-c-c)=B(c,c,c)$$

Fja F u tački B postiže USLOVNI MAKSIMUM

62. Naći ekstremume funkcije

$f(x,y) = x^2 + 4y^2$ uz uslov $x^2 + y^2 = 1$

$f(x,y)$, uslov $g(x,y)=0$

$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$

$$D = - \begin{vmatrix} 0 & g'_x & g'_y \\ g'_x & L''_{xx} & L''_{xy} \\ g'_y & L''_{yx} & L''_{yy} \end{vmatrix}$$

$D>0 \Rightarrow$ u M postiže minimum

$D<0 \Rightarrow$ u M postiže maksimum

$g(x,y) = x^2 + y^2 - 1$

uslov $g(x,y) = 0$

$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

$L(x,y,\lambda) = x^2 + 4y^2 + \lambda \cdot (x^2 + y^2 - 1)$

$\frac{\partial L}{\partial x} = 2x + 2\lambda x \qquad \frac{\partial L}{\partial x} = 0 \Rightarrow x(1+\lambda) = 0 \Leftrightarrow x = 0 \vee \lambda = -1$

$\frac{\partial L}{\partial y} = 8y + 2\lambda y \qquad \frac{\partial L}{\partial y} = 0 \Rightarrow y(4+\lambda) = 0 \Leftrightarrow y = 0 \vee \lambda = -4$

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1$ $\longrightarrow$ moguće kombinacije

$x=0 \quad \lambda = -4$

$x=0 \quad y=0$

$\lambda = -1 \quad y = 0$

~~$\lambda = -1 \quad \lambda = -4$~~ !

1° $x=0$; $y=0$

$0^2+0^2=1 \neq 0$ $\perp$

2° $x=0$; $\lambda=-4$

$y^2-1=0 \rightarrow y=\pm 1 \rightarrow \left.\begin{matrix} A(0,1) \\ B(0,-1) \end{matrix}\right\} \lambda=-4$

3° $y=0$; $\lambda=-1$

$x^2-1=0 \rightarrow x=\pm 1 \rightarrow \left.\begin{matrix} C(1,0) \\ D(-1,0) \end{matrix}\right\} \lambda=-1$

$\rightarrow$ A, B, C i D su stacionarne tačke

$\rightarrow$ našli stacionarne tačke $\rightarrow$ tražimo druge izvode, pravimo det. i gledamo $(>, <)\ 0$

$\frac{\partial^2 L}{\partial x^2} = 2+2\lambda \rightarrow L''_{xx}$

$\frac{\partial^2 L}{\partial x \partial y} = 0 \rightarrow L''_{xy}$

$\frac{\partial^2 L}{\partial y^2} = 8+2\lambda \rightarrow L''_{yy}$

$\frac{\partial g}{\partial x} = g'_x = 2x$ $\frac{\partial g}{\partial y} = g'_y = 2y$

$$D = -\begin{vmatrix} 0 & 2x & 2y \\ 2x & 2+2\lambda & 0 \\ 2y & 0 & 8+2\lambda \end{vmatrix}$$

Heseova determinanta

I $A(0,1)$

$\lambda=-4$

$2-8 \quad 8-8$

$$D = -\begin{vmatrix} 0 & 0 & 2 \\ 0 & -6 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -(-(-24)) = -24 < 0$$

USLOVNI MAKSIMUM

II $B(0,-1)$; $\lambda = -4$

$$D = -\begin{vmatrix} 0 & 0 & -2 \\ 0 & -6 & 0 \\ -2 & 0 & 0 \end{vmatrix} = -(-(-24)) = -24 < 0$$

$\Rightarrow$ u tački $B(0,-1)$ f-ja dostiže uslovni maksimum

III $C(1,0)$; $\lambda = -1$

$$D = -\begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{vmatrix} = -(-24) = 24 > 0$$

$\Rightarrow$ u tački $C(1,0)$ f-ja postiže uslovni minimum

IV $D(-1,0)$; $\lambda = -1$

$$D = -\begin{vmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 6 \end{vmatrix} = -6 \cdot (-4) = 24 > 0$$

$\Rightarrow$ u tački $D(-1,0)$ f-ja postiže uslovni ~~maks~~ minimum

Kad imamo 3. promjenljive: Oivičeni Hese

Posmatramo 2 determinante

$$D_4 = -\begin{vmatrix} 0 & g'_x & g'_y & g'_z \\ g'_x & L''_{xx} & L''_{xy} & L''_{zx} \\ g'_y & L''_{xy} & L''_{yy} & L''_{zy} \\ g'_z & L''_{xz} & L''_{yz} & L''_{zz} \end{vmatrix}$$

$$D_3 = \begin{cases} -\begin{vmatrix} 0 & g'_x & g'_y \\ g'_x & L''_{xx} & L''_{yx} \\ g'_y & L''_{xy} & L''_{yy} \end{vmatrix}, & g'_x \neq 0 \ \vee \ g'_y \neq 0 \\ -\begin{vmatrix} 0 & g'_y & g'_z \\ g'_y & L''_{yy} & L''_{zy} \\ g'_z & L''_{yz} & L''_{zz} \end{vmatrix}, & g'_x = 0 \ ; \ g'_y = 0 \end{cases}$$

1) $D_3 > 0 \wedge D_4 > 0 \Rightarrow$ MINIMUM

2) $D_3 < 0 \wedge D_4 > 0 \Rightarrow$ MAKSIMUM

3) $D_4 < 0 \Rightarrow$ NEMA EKSTREMUM

D_4 uvijek isto a D_3 zavisi od g'_x i g'_y

~~V~~ da li je oba ili jedan

~~3 stac. tačke ; 1 → $D_4 < 0$ → nema ekstrem.~~

63. $f(x,y,z) = x^2 + y^2 + z^2$

$2 = z - xy$

$g(x,y,z) = xy - z + 2$

uslov $g(x,y,z) = 0$

1) Klasično Lagranž
2) Silvesterov krit. al za L!!!
3) uvrstimo uslov
4) ovaj posl. način

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot g(x,y,z)$$

$$L = x^2 + y^2 + z^2 + \lambda\,(xy - z + 2)$$

$$\frac{\partial L}{\partial x} = 2x + \lambda y \rightarrow \frac{\partial L}{\partial x} = 0 \Rightarrow 2x + \lambda y = 0 \Rightarrow x = -\frac{1}{2}\lambda y$$

$$\frac{\partial L}{\partial y} = 2y + \lambda x \rightarrow \frac{\partial L}{\partial y} = 0 \Rightarrow 2y + \lambda x = 0$$

$$\frac{\partial L}{\partial z} = 2z - \lambda \rightarrow \frac{\partial L}{\partial z} = 0 \Rightarrow z = \frac{1}{2}\lambda$$

$$\frac{\partial L}{\partial \lambda} = xy - z + 2 \rightarrow \frac{\partial L}{\partial \lambda} = 0 \Rightarrow xy - z + 2 = 0$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow 2y + \lambda x = 0 \rightarrow 2y - \frac{1}{2}\lambda^2 y = 0$$

$$y\left(2 - \frac{1}{2}\lambda^2\right) = 0$$

$$\Rightarrow y = 0 \quad \vee \quad 2 - \frac{1}{2}\lambda^2 = 0 \rightarrow \lambda = \pm 2$$

$$4 - \lambda^2 = 0$$

za $\lambda = -2 \rightarrow x = y$

Za $\Omega = -2 \rightarrow x = y$

Kako tražim stacionarne tačke?

AGATHA RUIZ DE LA PRADA

64) Naći najveću i najmanju vrijednost funkcije $z = x^2 + y^2$ u oblasti:

$$D = \{(x,y) \mid x^2 + y^2 - 2\sqrt{2}x - 2\sqrt{2}y - 5 \leq 0\}$$

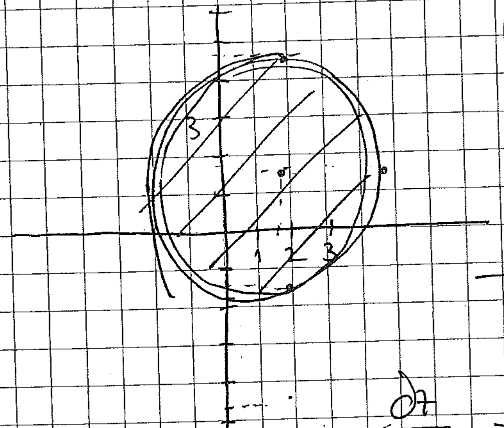
→ ne treba 2. izvod → tražimo globalni a ne lokalni ekstremum → nađemo vrijednosti, uporedimo i kažemo max/min

$$D = \{(x,y) \mid x^2 + y^2 - 2\sqrt{2}x - 2\sqrt{2}y - 5 \leq 0$$

$$x^2 - 2\sqrt{2}x + (\sqrt{2})^2 + y^2 - 2\sqrt{2}y + 2 - 4 - 5 \leq 0$$

$$(x-\sqrt{2})^2 + (y-\sqrt{2})^2 \leq 9$$

→ $(x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9$ → Jna kruga sa centrom u $C(\sqrt{2},\sqrt{2})$ i poluprečnikom r



(1) Stacionarne tačke u unutrašnjosti: izjednačavamo parcijalne izvode sa 0

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial x} = 0 \Leftrightarrow x = 0$$

$$\frac{\partial z}{\partial y} = 2y \qquad \frac{\partial z}{\partial y} = 0 \Leftrightarrow y = 0$$

$O(0,0) \in$ unutrašnjosti oblasti D

(2) posmatrajmo granicu oblasti D

njena jednačina: $(x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9$

Lagranžova funkcija i tražimo stacionarne tačke

$$L(x,y,\lambda) = x^2+y^2 + \lambda\left((x-\sqrt{2})^2+(y-\sqrt{2})^2-9\right)$$

$$\frac{\partial L}{\partial x} = 2x + 2\lambda(x-\sqrt{2})$$

$$\frac{\partial L}{\partial y} = 2y + 2\lambda(y-\sqrt{2})$$

$$\frac{\partial L}{\partial \lambda} = (x-\sqrt{2})^2 + (y-\sqrt{2})^2 - 9$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow (1+\lambda)x - \lambda\sqrt{2} = 0 \Rightarrow x = \frac{\lambda\sqrt{2}}{1+\lambda}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow (1+\lambda)y - \lambda\sqrt{2} = 0 \Rightarrow y = \frac{\lambda\sqrt{2}}{1+\lambda}$$

$x=y$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow (x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 9$$

$$x^2 - 2\sqrt{2}x + 2 + y^2 - 2\sqrt{2}y + 2 - 9 = 0$$

$$2x^2 - 4\sqrt{2}x - 5 = 0 \rightarrow x_{1,2} = \frac{-4\sqrt{2} \pm 6\sqrt{2}}{4}$$

$$x_1 = \frac{5\sqrt{2}}{2} \rightarrow y_1 = x_1 \ (A); \quad x_2 = -\frac{\sqrt{2}}{2}; \ y_2 = x_2 \ (B)$$

$$z(0) = z(0,0) = 0^2 + 0^2 = 0$$

$$z(A) = z\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right) = 25$$

$$z(B) = z\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2}+\frac{1}{2} = 1$$

→ u tački O fja $z(x,y)$ postiže najmanju vrijednost na oblasti D, a to je $z_{min} = 0$

→ u tački A postiže najv. vr. na D a to je $z_{max} = 25$

AGATHA RUIZ DE LA PRADA

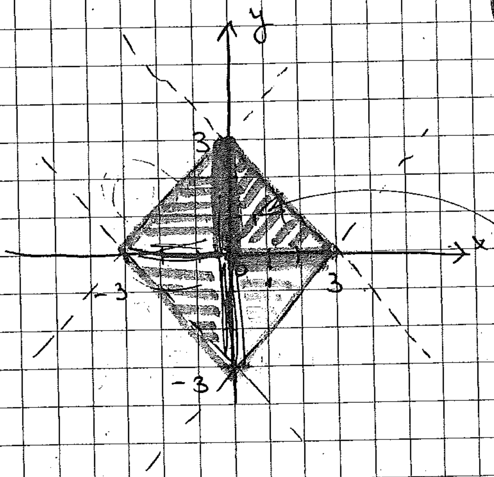
65. Odrediti najmanju i najveću moguću vrijednost fje $z(x,y) = x^2 - xy + y^2$ u oblasti:

$D = \{ (x,y) \mid |x| + |y| \leq 3 \}$

na osnovu čega $\geq$ ili samo $>$?

1) analiza funkcije $|x|+|y|$

$$|x| + |y| = \begin{cases} x+y, & x \geq 0, y \geq 0 \\ -x+y, & x<0, y \geq 0 \\ x-y, & x \geq 0, y<0 \\ -x-y, & x<0, y<0 \end{cases}$$



Baratati dobro granicama ?

$D = \{ (x,y) \mid (x+y \leq 3 \wedge x \geq 0 \wedge y \geq 0)$
$\vee (-x+y \leq 3 \wedge x<0 \wedge y \geq 0)$
$\vee (x-y \leq 3 \wedge x \geq 0 \wedge y<0)$
$\vee (-x-y \leq 3 \wedge x<0 \wedge y<0) \}$

1 $x+y=3$
$y=3-x$
$x+y \leq 3$
$y \leq 3-x$

2 $-x+y=3$
$y=3+x$
$-x+y \leq 3$
$y \leq 3+x$

3 $x-y=3$
$y=x-3$
$y \geq x-3$

4 $-x-y=3$
$y=-3-x$
$-x-y \leq 3$
$y \geq -x-3$

D je zatvorena i ograničena oblast pa fja z na njoj postiže i najmanju i najveću vrijednost.

I UNUTRAŠNJOST

$$\frac{\partial z}{\partial x} = 2x-y \rightarrow \frac{\partial z}{\partial x} = 0 \Rightarrow 2x-y=0 \quad \Big) \cdot 2$$

$$\frac{\partial z}{\partial y} = -x+2y \rightarrow \frac{\partial z}{\partial y} = 0 \Rightarrow -x+2y=0$$

$x=0, \; y=0$

$O(0,0) \in$ unutrašnjosti

II Granice postavljamo 4 Lagranžove funkcije

$1^\circ \quad z(x,y) = x^2 - xy + y^2, \quad y = -x+3$

$0 \leq x \leq 3$

→ zadatak iz uslovnog $\quad x \in [0,3]$

ekstremuma; koristimo uslov $y = -x+3$

$f_1(x) = f(x, -x+3) = x^2 - x(-x+3) + (-x+3)^2,$

$x \in [0,3]$

→ $f_1(x) = 3x^2 - 9x + 9, \quad x \in [0,3]$

→ Kao da tražimo tačke u kojima $f_1(x)$ može postići globalni ekstremum

$f_1'(x) = 0 \iff 6x - 9 = 0$

$x = \frac{3}{2}$

→ za fju f_1 uočavamo tačke:

$x = 0; \quad x = \frac{3}{2}; \quad x = 3$

→ za fju f uočavamo tačke

$A_1(0,3); \quad A_2\left(\frac{3}{2}, \frac{3}{2}\right); \quad A_3(3,0)$

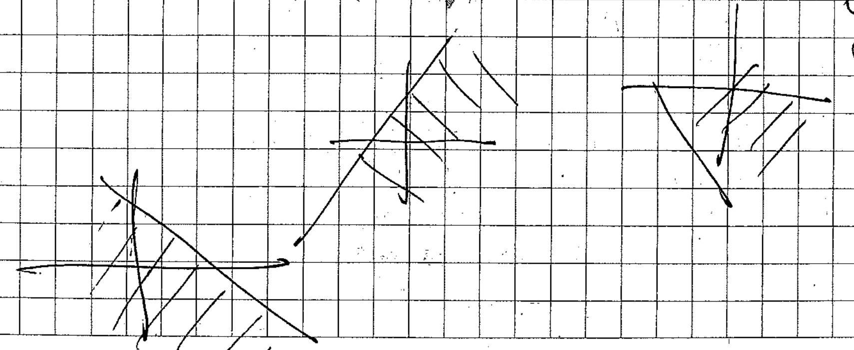
→ slično dobijamo još 5 tačaka:

$A_4(0,3); \quad A_5\left(-\frac{3}{2}, \frac{3}{2}\right); \quad A_6(-3,0); \quad A_7\left(-\frac{3}{2}, -\frac{3}{2}\right),$

$A_8\left(\frac{3}{2}, -\frac{3}{2}\right)$

→ računamo vrijednosti fje u svim tačkama i uzmemo max/min

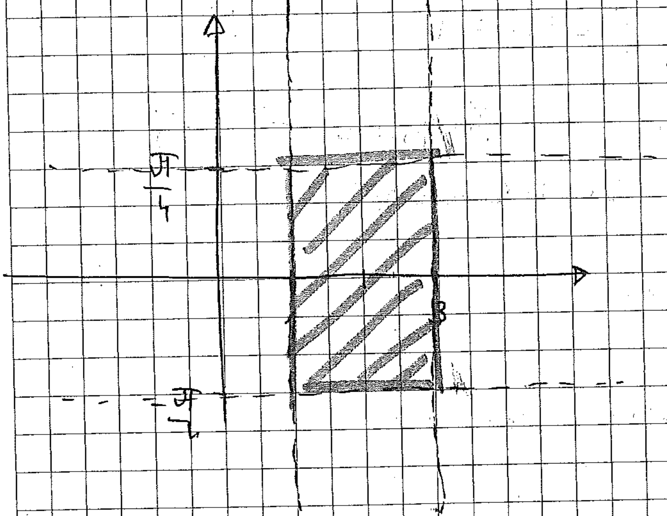
$-x - y \leq 3$



66. Data je f-ja $f(x,y) = (4x - x^2) \cdot \cos y$ i skup $D = \{ (x,y) \mid 1 \le x \le 3 \land -\frac{\pi}{4} \le y \le \frac{\pi}{4} \}$.

a) Odrediti lokalne ekstremume f-je f u unutrašnjosti oblasti D → lokalni

b) Odrediti najveću i najmanju vr. f-je f na oblasti D → globalni ∥ ekstremne vr. f-je



a) $\frac{\partial f}{\partial x} = \cos y \cdot (4 - 2x)$

$\frac{\partial f}{\partial y} = (4x - x^2) \cdot (-\sin y)$

$\frac{\partial f}{\partial x} = 0 \Leftrightarrow \cos y (4 - 2x) = 0$

$4 - 2x = 0$

$2x = 4 \rightarrow x = 2$

za $\cos y = 0$ ne pripada domenu

$\frac{\partial f}{\partial y} = 0 \Leftrightarrow -(4x - x^2) \sin y = 0 \rightarrow -x \cdot \sin y \cdot (4 - x)$

$x = 0 \lor x = 4 \lor y = 0$

nije iz domena; jer nije iz oblasti

~~da su npr sve ove tačke iz~~ unutrašnjosti → kombinacije (2,0) (0,0) (4,0) ?

$A(2, 0) \in$ unutrašnjost ; $(0,0) \notin$ unutrašnjost

1° $\Delta_1 = \frac{\partial^2 f}{\partial x^2}(A)$

2° $\Delta_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(A) & \frac{\partial^2 f}{\partial x \partial y}(A) \\ \frac{\partial^2 f}{\partial y \partial x}(A) & \frac{\partial^2 f}{\partial y^2}(A) \end{vmatrix}$

b) I unutrašnjost

parc. izvodi i izjednačeni

→ urađeno pod a) → $A(2,0)$

II granice

4 djela

$f(x,y) = (4x-x^2)\cdot \cos y$

$y = -\frac{\pi}{4}, \quad x \in [1,3]$

$f_1(x) = f(x, -\frac{\pi}{4}) = (4x-x^2)\cdot \frac{\sqrt{2}}{2}, \quad x \in [1,3]$

→ na ovaj način sva 4 djela

$A_1(1, -\frac{\pi}{4}) \quad B_1(2, \frac{\pi}{4}) \quad A_3(3, -\frac{\pi}{4})$

III $f(x,y) = (4x-x^2)\cdot \cos y, \quad x=3; \ y \in [-\frac{\pi}{4}, \frac{\pi}{4}]$

$f_2(y) = f(3,y) = 3\cdot \cos y, \quad y \in [-\frac{\pi}{4}, +\frac{\pi}{4})$

$f_2'(y) = 0$

...

---

67. Odrediti ekstremume f-je $u(x,y,z) = x^2+2y^2-z^2$ na skupu $D = \{(x,y,z) \mid x^2+y^2 \le 1, \ x+z=2\}$

→ uvrstimo uslov $x+z=2 \to z = 2-x$

$\Rightarrow f(x,y) = u(x,y,2-x)$

→ tražimo extr. f-je $f(x,y)$

$D_1 = \{(x,y) \mid x^2+y^2 \le 1\}$

POVRŠI DRUGOG REDA

68. Naći centar i poluprečnik sfere

$$x^2+y^2+z^2-3x+5-4z=0$$

$$x^2-3x+\frac{9}{4}+y^2+z^2-4z+4-4+5\cancel{-0}-\frac{9}{4}=0$$

$2\cdot\frac{3}{2}x=3x$

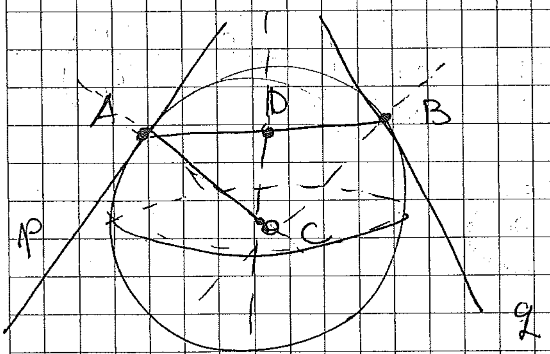
$$\left(x-\frac{3}{2}\right)^2+y^2+(z-2)^2=\frac{5}{4}$$

$\rightarrow$ centar $C\left(\frac{3}{2},0,2\right)$; $r=\frac{\sqrt{5}}{2}$

69. Naći jednačinu sfere koja dodiruje pravu

$p: \frac{x-1}{3}=\frac{y+4}{6}=\frac{z-6}{4}$ u tački $A(1,-4,6)$

i pravu $q: \frac{x-4}{2}=\frac{y+3}{1}=\frac{z-2}{-6}$ u tački $B(4,-3,2)$



α: ravan koja sadrži A i ortogonalna je na p.

β: ravan koja sadrži B i ortogonalna je na q

γ: ravan koja sadrži D (središte AB) i $\perp\ \ell(AB)$

$\rightarrow$ u presjeku ovih ravni C

$\alpha\cap\beta\cap\gamma=\{C\}$ $\rightarrow$ centar sfere

$p: \underbrace{\frac{x-1}{3}=\frac{y+4}{6}=\frac{z-6}{4}}$

$\vec{s_p}=(3,6,4)$

α: $A(4,-4,6)$

$\vec{n_\alpha}=\lambda\cdot\vec{s_p}$

$n_\alpha=1\cdot(3,6,4)=(3,6,4)$